CHARACTERISTICS OF VORTEX STABILIZATION OF AN ELECTRICALLY CONDUCTING GAS IN A RADIAL ELECTRIC FIELD

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The authors of paper [1] investigated the instability of the surface separating a heavy electrically conducting fluid from a lighter nonconducting fluid situated above it in the strong electric field of a plane electrode. It was shown that the electrically conducting fluid formed a circuit with the electrode as the result of the development of surface instability. In this paper we apply the mathematical methods and results of [1] to an investigation of vortex stabilization of a plasma filament in a radial electric field. Formulas are obtained for calculation of the minimum velocity of rotation of the medium necessary for stabilization of the filament as well as of the critical potential for which the filament surface becomes unstable (hydrodynamic instability which is not connected with the electric field is not treated or taken into account). It is shown that under specific conditions this critical potential is considerably lower than the avalanche-breakdown potential and that the mechanism by which the plasma forms a circuit with the electrode can play an important part in vortex stabilization.

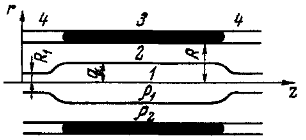

Fig. 1

Statement of the problem. Let a long plasma filament of radius R_1 be situated inside a cylindrical tube of radius R, shown in Fig. 1, where 1 is the plasma filament, 2 is the stabilizing medium, 3 is the electrode, and 4 is the tube. The tube consists of two halves between which is a cylindrical electrode, of inside radius R, with round ends. The length of the electrode is considerably greater than R. The potentials of the tube and the plasma are both zero, while the potential of the electrode is U_p. The axisymmetric position of the plasma filament is stabilized because of the rotation of the plasma and the denser nonconducting medium between the plasma and the electrode. The densities of the plasma and the medium are ρ_1 and ρ_2, respectively, and their velocities of rotation v for $r \geq R_1$ obey the law

$$vr = A, \qquad A = \text{const}. \tag{1}$$

The radius of the filament inside the electrode increases to a under the influence of the electric field. Under these conditions the field between the cylindrical surfaces $r = a$ and $r = R$ inside the electrode and far from its ends can be taken as cylindrically symmetrical until the appearance of instability. We wish to determine the least potential U_p for which the plasma filament becomes unstable, and the minimum circulation $2\pi A$ necessary for its stabilization. To determine these quantities, we consider two cases of equilibrium of the boundary between the plasma and the nonconducting medium.

Case 1. The nonconducting medium is a liquid with a surface-tension coefficient Ω. Let the plasma surface be given a small perturbation which slightly alters the surface curvature $r = a$. The perturbed surface f is described by the function $r = f(z)$ and satisfies the equation

$$\frac{d^2}{dz^2} \ln \frac{f}{a} + \lambda^2 \ln \frac{f}{a} = 0 \qquad \left(\begin{matrix} f = a(1+\psi) \\ \psi \ll 1 \end{matrix} \right). \tag{2}$$

The potential distribution inside the nonconducting liquid satisfies the Laplace equation

$$\Delta U(r, z) = 0. \tag{3}$$

In this region U is represented as

$$U = \frac{U_p}{\ln k} \left[\ln \frac{r}{a} - \frac{I_0(\lambda r) K_0(\lambda R) - I_0(\lambda R) K_0(\lambda r)}{m} \ln \frac{f}{a} \right]$$
$$\left(k = \frac{R}{a} \right), \tag{4}$$

$$m = I_0(\lambda a) K_0(\lambda R) - I_0(\lambda R) K_0(\lambda a). \tag{5}$$

Here I_ν and K_ν are Bessel functions of an imaginary argument of order ν.

Function (4) satisfies Eq. (3), boundary condition $U(R, z) = U_p$, and also (approximately) boundary condition $U(f, z) = 0$. The forces acting per unit area of the plasma surface are the electric force F_e, the surface-tension force F_Ω, and the static pressure differential between the liquid and the plasma F_p. The equilibrium condition is

$$F_e = F_\Omega + E_p. \tag{6}$$

The electrical force is determined by the equation

$$F_e = \frac{\varepsilon}{8\pi} (\operatorname{grad} U)_f^2, \tag{7}$$

where ε is the dielectric constant of the liquid. Taking (2) into account, we have from (4) and (7)

$$F_e \approx \frac{\varepsilon U_p^2}{8\pi \ln^2 k} \left(\frac{1}{f^2} - \frac{2\lambda n}{fm} \ln \frac{f}{a} \right),$$
$$n = I_1(\lambda a) K_0(\lambda R) + I_0(\lambda R) K_1(\lambda a). \tag{8}$$

The component of force F_p is given by

$$F_p = (p_3 - p_2) - (p_1 - p_0) - \frac{\Omega}{R_1}. \tag{9}$$

Here p_0 and p_1 are the static pressures in the plasma for $r = (R_1 - 0)$ and $r = (f - 0)$, respectively, while p_2 and p_3 are the static pressures in the liquid for $r = (R_1 + 0)$ and $r = (f + 0)$.

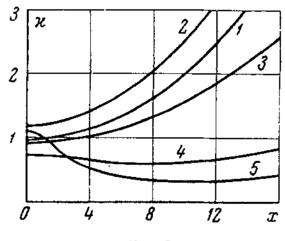

Fig. 2

We can determine $(p_3 - p_2)$ and $(p_1 - p_0)$ from the momentum equation

$$\frac{dp}{dr} = \frac{\rho v^2}{r}. \tag{10}$$

Taking (1) into account and integrating (10), we have from (9)

$$F_p = \frac{(\rho_2 - \rho_1) A^2}{2f^2} \left(\frac{f^2}{R_1^2} - 1 \right) - \frac{\Omega}{R_1}. \tag{11}$$

The surface tension is given by the equation

$$F_\Omega = \Omega (c_1 + c_2). \tag{12}$$

Here c_1 and c_2 are the greatest and least surface curvatures. The curvature will be taken as positive if the axis of the plasma filament lies on its concave side. Since the change of surface curvature is small, the quantity f_z^2 may be neglected as small in comparison with unity. When this is taken into account we have

$$c_1 + c_2 = \frac{1}{f} - f_{zz} \,. \tag{13}$$

Allowing for the fact that ψ is small we expand $\ln(f/a)$ in series and retain the first term

$$\ln \frac{f}{a} = \psi, \qquad f_{zz} = a\psi_{zz} \,. \tag{14}$$

Substituting (14) into (13) and taking (2) into account we have

$$F_\Omega = \Omega\,(1/f + a\lambda^2\psi). \tag{15}$$

Substituting (8), (11), and (15) into (6), we obtain the linearized equation

$$\frac{\varepsilon U_p^2}{8\pi \ln^2 k}\left(1 - \frac{2a\lambda n}{m}\psi\right) = \frac{(\rho_2-\rho_1)A^2}{2}\left[\frac{a^2}{R_1^2}(1+2\psi) - 1\right] + \Omega\left[a(1+\psi) + a^3\lambda^2\psi - \frac{a^2}{R_1}(1+2\psi)\right]. \tag{16}$$

Equation (16) is satisfied if

$$\frac{\varepsilon U_p^2}{8\pi \ln^2 k} = \frac{(\rho_2-\rho_1)A^2}{2}\left(\frac{a^2}{R_1^2} - 1\right) + a\Omega\left(1 - \frac{a}{R_1}\right), \tag{17}$$

$$-\frac{\varepsilon U_p^2}{4\pi \ln^2 k}\,\frac{a\lambda n}{m} = (\rho_2-\rho_1)A^2\frac{a^2}{R_1^2} + a\Omega\left(1 + a^2\lambda^2 - \frac{2a}{R_1}\right). \tag{18}$$

We now introduce the dimensionless quantities

$$u = \frac{\varepsilon U_p^2}{4\pi(\rho_2-\rho_1)A^2}, \qquad w = \frac{a\Omega}{(\rho_2-\rho_1)A^2},$$

$$x = a\lambda, \qquad y = \frac{a}{R_1}, \qquad \varphi_1 = \frac{m}{a\lambda n} \,. \tag{19}$$

When (19) is taken into account, Eqs. (17) and (18) can be rewritten in the form

$$u = [(y^2-1) + 2w(1-y)]\ln^2 k,$$
$$u = -\varphi_1[y^2 + w(1 + x^2 - 2y)]\ln^2 k \,. \tag{20}$$

On combining these relations we have

$$y^2 - 2wy = \frac{1 - 2w - w\varphi_1(1+x^2)}{1+\varphi_1},$$
$$u = \varphi_1\frac{w - 1 - wx^2}{1+\varphi_1}\ln^2 k \,. \tag{21}$$

Figure 2 gives u as a function of x when $w = 10^{-2}$: curve 1 is for $\varkappa = 10^5 u$, $k = 0.01$; curve 2 is for $\varkappa = 10^4 u$, $k = 1.05$; curve 3 is for $\varkappa = 10^3 u$, $k = 1.1$; curve 4 is for $\varkappa = 10^2 u$, $k = 1.2$; curve 5 is for $\varkappa = 10u$, $k = 1.5$.

It is clear from the graphs that there is a minimum in the function $u = u(x)$. This minimum corresponds to the least value of u for which neutral equilibrium is possible. The value of x for which the minimum occurs is given by the equation

$$\frac{d}{dx}\ln\frac{\varphi_1}{1+\varphi_1} = \frac{2wx}{w - 1 - wx^2} \,. \tag{22}$$

The required quantities are calculated in the following order. Equation (22) is used to find x for given values of k and w. The quantities y and u are then determined from Eqs. (21). Next, for given R and Ω we find

$$a = \frac{R}{k}, \qquad R_1 = \frac{a}{y}, \qquad A^2 = \frac{a\Omega}{(\rho_2-\rho_1)w},$$
$$U_p^2 = \frac{4\pi(\rho_2-\rho_1)A^2 u}{\varepsilon} \,.$$

The curves and formulas obtained show that the quantity U_p decreases rapidly as k and A decrease.

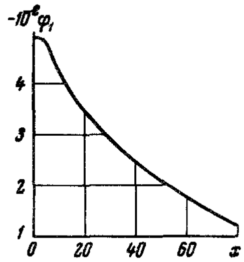

Fig. 3

Case 2. The nonconducting medium is a gas whose density increases linearly from ρ_1 when $r = R_1$ (far from the electrode) to ρ_R when $r = R$. If the density of the nonconducting gas varies according to the law

$$\rho_2 = \rho_1[1 + \beta(r - R_1)], \qquad \beta = \frac{\rho_R - \rho_1}{\rho_1(R - R_1)}, \tag{23}$$

far from the electrode where the effect of the electric field can be neglected, Eqs. (1) and (10) can be used to obtain the following formula for F_p.

$$F_p = \frac{\beta\rho_1 A^2}{2f^2}\left[\frac{f}{R_1} - 2f + R_1\right]. \tag{24}$$

Substituting (8) and (24) into (6) and introducing the dimensionless quantity

$$\theta = \frac{\varepsilon U_p^2}{4\pi\beta\rho_1 A^2 a}, \tag{25}$$

we obtain

$$a\theta\left(1 - \frac{2\lambda n a}{m}\psi\right) = \left[\frac{a^2}{R_1}(1+2\psi) - 2a(1+\psi) + R_1\right]\ln^2 k. \tag{26}$$

The linearized equation (26) is satisfied if

$$\theta = \frac{(y-1)^2}{y}\ln^2 k, \qquad \theta = (1-y)\varphi_1\ln^2 k \,. \tag{27}$$

A graph of the function

$$\varphi_1 = \frac{I_0(x)K_0(kx) - I_0(kx)K_0(x)}{x[I_1(x)K_0(kx) + I_0(kx)K_1(x)]}, \tag{28}$$

introduced previously, is given in Fig. 3 for $k = 1.05$.

We now give a series of quantities φ_1 for certain values of x:

$x =$	0	0.5	1	2	5	10	20	30	40	60
$-10^2\varphi_1 =$	4.88	4.94	4.87	4.85	4.78	4.50	3.34	2.95	2.40	1.65

With asymptotic expansions of the Bessel functions, it can be shown that for large values of x the function φ_1 can be represented in the form

$$\varphi_1 = \frac{\mathrm{th}\,(1-k)\,x}{x} \,. \tag{29}$$

It is clear from (29) that $\varphi_1 \to 0$ as $x \to \infty$. However, for large x (or for large λ, which is the same) the linear approximation is insufficient, since for large λ the assumption that the change of surface curvature is small is not valid. Nevertheless, judging from the form of the function φ_1, we can conclude that the critical potential U_p can have different values depending on the form of surface perturbation of the plasma filament for relatively large values of x. This difference from case 1 is explained by the absence of surface-tension forces which increase the stability of the plasma filament when it suffers local deformation.

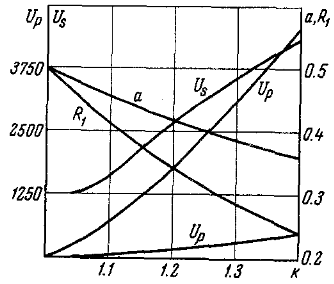

Fig. 4

For $x \ll 1$ we have

$$\varphi_1 \approx \left(\frac{2x^2 \ln k}{4+x^2} - 1\right)^{-1} \ln k\ . \tag{30}$$

Thus it is clear that for x = 0 the function given by the second equation of (27) has a minimum equal to

$$\theta = (y-1)\ \ln^3 k\ . \tag{31}$$

From (31) and the first equation of (27) we have

$$\ln k = (y-1)y^{-1}, \qquad \theta = (y-1)^4 y^{-3}\ . \tag{32}$$

The required quantities are calculated in the following order.

For a known value of R we specify y and find k from (32). Then we determine $a = R/k$, $R_1 = a/y$ and find U_p from the formula

$$U_p = \frac{2(a-R_1)^2 A}{a}\left(\frac{\pi\beta\rho_1}{\varepsilon R_1}\right)^{0.5}\ . \tag{33}$$

The potential found is the least potential for which neutral equilibrium is possible if the perturbed surface is cylindrical. This assertion results from the first equation of (27), which is the equilibrium condition for a cylindrical surface; if we introduce the new variable $\theta_1 = a\theta$ and set $d\theta_1/da = 0$, we obtain formula (32) and consequently formula (33) as well.

Results for U_p and a, calculated from formulas (32) and (33) for $\varepsilon = 1$, R = 0.5 cm, $\rho_1 = 1.407 \cdot 10^{-4}$ g·cm^{-3}, and $\rho_R = 8.86 \cdot 10^{-4}$ g·cm^{-3}, are given in Fig. 4, where the lower and upper curves for U_p correspond, respectively, to $A = 10^2$ and $A = 10^3$ cm^2/sec. It is clear that as R_1 increases, U_p rapidly decreases to very small values. The dependence of U_p on gas circulation, gas density, and other quantities is immediately apparent from formula (33). Formula (33) shows that as A decreases (i.e., the rotation velocity of the gas and the tube radius decrease), the stabilizing effect of rotary motion on the plasma filament decreases, with other conditions remaining the same. The graphs show that as filament radius R_1 decreases, the stabilizing effect of the gas rapidly increases, i.e., U_p increases sharply. This is associated with the increase of gas velocity for a decrease of radius in accordance with (1). In actual cases of vortex stabilization of a plasma filament the gas velocity close to the axis decreases due to the gas viscosity. Thus, for small R_1 the value of U_p must be less than that given by the formulas quoted above. Moreover, as a result of friction, the gas velocity close to the electrode wall decreases and is zero at the wall itself. Accordingly, the stabilizing effect of the gas and the value of U_p should be very much decreased if the plasma filament is close to the wall. Formulas which take these characteristics into account can be obtained in the same way if the law governing the change of gas velocity over the channel radius is given.

Plasma breakdown. In continuing the analogy with the results of [1], we can expect the following behavior of the plasma filament after the onset of instability. The distance between the electrode and the filament decreases as the result of surface deformation of the latter. This leads to a further increase of the electric-field strength in the deformation zone, and this gives rise to progressive development of instability until the plasma forms a circuit with the electrode. This phenomenon can be called plasma breakdown, as distinct from the normal avalanche or streamer breakdown. If there are no other competing processes and if U_p is less than the normal breakdown potential U_s, plasma breakdown occurs first and there is no normal breakdown. To compare the quantities U_p and U_s we calculate U_s from Townsend's theory for the case in which the electrode is the cathode, and the plasma is the anode.

The breakdown potential for the gap between the cylindrical surfaces $r = a$ and $r = R$ is determined by the formula

$$\ln\left(1+\frac{1}{\gamma}\right) = \int_a^R \alpha dr\ . \tag{34}$$

Here α and γ are Townsend's first and second ionization coefficients.

Allowing for the temperature dependence, we can express α as follows:

$$\alpha = \frac{M_1 p}{t}\exp\left(-\frac{B_1 p}{Et}\right). \tag{35}$$

Here E is the electric field strength, T is the gas temperature, and T_0 is the temperature for which the coefficients M_1 and B_1 are determined experimentally.

The following values were assumed in the calculations: p = = 10 N·cm^{-2}, T = 2500°K for the temperature of the filament surface, T_R = 400°K for the gas temperature at the tube wall, T_0 = 300°K, and finally a temperature variation according to the law

$$t = s_0 - s\zeta, \qquad s = \frac{T_1 - T_R}{T_0(k-1)}, \qquad s_0 = s + \frac{T_1}{T_0}, \qquad \zeta = \frac{r}{R}.$$

This corresponds to a linear increase of air density over the radius. With the values of M_1 and B_1 given in [2], we have $M = M_1 p = 11\,400$, $B = B_1 p = 277\,400$ for p = 10 N·cm^{-2}. We can assume $\gamma = 10^{-2}$ for air [2]. By means of the symbols introduced, expression (34) can be reduced to the form

$$\ln\left(1+\frac{1}{\gamma}\right) = \int_1^k \frac{M}{t}\exp\left(-\frac{\zeta B \ln k}{tU_s}\right) d\zeta\ . \tag{36}$$

Figure 4 gives some results for U_s calculated from formula (36). Clearly, for small k and A the value of U_p is considerably lower than U_s. Thus, we can expect that under these circumstances the plasma makes contact with the electrode before normal breakdown occurs due to instability. For higher gas velocities and larger values of k, U_p becomes larger than U_s. In this case normal electrical breakdown is more probable.

The calculations given above show that under specific conditions the surface instability of a plasma filament may play an important part in breakdown between the plasma and a cold electrode.

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